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Towards a Fermi liquid theory of the $\nu = \frac{1}{2}$ state: magnetized composite fermions

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Abstract. The fermionic Chern–Simons approach has had remarkable success in the description of quantum Hall states at even-denominator filling fractions $\nu = 1/2m$. In this paper we review a number of recent works concerned with modelling this state as a Landau–Silin Fermi liquid. We will then focus on one particular problem with constructing such a Landau theory that becomes apparent in the limit of high magnetic field, or equivalently the limit of small electron band mass m_b. In this limit, the static response of electrons to a spatially varying magnetic field is largely determined by kinetic energy considerations. We then remedy this problem by attaching an orbital magnetization to each fermion to separate the current into magnetization and transport contributions, associated with the cyclotron and guiding centre motions respectively. This leads us to a description of the $\nu = 1/2m$ state as a Fermi liquid of magnetized composite fermions which correctly predicts the m_b-dependence of the static and dynamic response in the limit $m_b \to 0$. As an aside, we derive a sum rule for the Fermi liquid coefficients for the Chern–Simons Fermi liquid. This paper is intended to be readable by people who may not be completely familiar with this field.

1. Introduction

The Chern–Simons (or 'composite') fermion theory has had a number of remarkable successes in the description of quantum Hall states [1, 2]. Based on the work of Jain [3], and Zhang, Hansson, and Kivelson [4], the Chern–Simons fermion picture was first introduced by Lopez and Fradkin [5] to study incompressible fractional quantized Hall states. Later, in work by Halperin, Lee and Read (HLR) [1], as well as Kalmeyer and Zhang [6], the theory was used to study even-denominator filling fractions. A prediction of this approach is that the states at even-denominator filling fraction should be compressible Fermi-liquid-like states. However, several major problems have appeared in describing these states as Fermi liquids. Many of these problems are related to the infra-red divergent properties of the Chern–Simons gauge-field fluctuations [1, 7, 8, 9, 10]. Recently, it has been pointed out that there are also complications that are unrelated to infra-red properties [11]. These complications become most pronounced in the limit of large magnetic field (or equivalently when the electron band mass m_b is taken to zero). A resolution to the $m_b \to 0$ problems has been proposed in reference [11] which involves binding of magnetization (unrelated to spin) to each Chern–Simons quasiparticle. The resulting magnetized Fermi liquid description of even-denominator Hall states yields the correct behaviour in the $m_b \to 0$ limit.

The current paper is written mainly to make the work of reference [11] more accessible to those who are not experts in the field. Thus, much background material will be discussed

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in detail. In section 2 a brief review is given of previous works relating to the Chern–Simons fermionic picture of fractional Hall states. We begin by reminding the reader of a few essentials of quantum Hall physics in section 2.1. In section 2.2 the basic Chern–Simons transformation is described in detail and in section 2.3 the Chern–Simons mean-field description of both the incompressible fractional Hall states and the compressible even-denominator states is discussed. Section 2.4 is devoted to a brief review of several of the attempts to perform a controlled perturbation theory around this mean-field solution. We will also briefly mention some of the works that focus on the infra-red divergences related to the gauge-field fluctuations.

In section 2.5 we define and discuss the electromagnetic response functions K and related response functions which are the objects that we will attempt to calculate throughout the rest of the paper. The simplest and most commonly used approximation (beyond mean field) for calculating these response functions is the random-phase approximation (RPA). This approximation will be discussed in section 2.6. It is pointed out that this approximation either breaks Galilean invariance or incorrectly describes the energy scale of the low-energy excitations. We then discuss how this problem is corrected by using the modified RPA (MRPA) from reference [13].

In section 3 we discuss the physics of the large-magnetic-field (or $m_b \to 0$) limit. In particular, in section 3.1 we focus on the zero-frequency, finite-wavevector electromagnetic response in this limit. We show that the (M)RPA incorrectly models some features of this response. In section 3.2 we propose that these problems can be repaired by binding magnetization to each Chern–Simons quasiparticle. In essence, this binding allows for a separation of the current into a magnetization current which is associated with the cyclotron motion of electrons and a transport current associated with the guiding centre motion. Following reference [11], in section 3.3 a 'magnetized modified RPA' (M^2RPA) is defined that uses this magnetization binding approach in combination with the MRPA to calculate the physical electromagnetic response function K.

Section 4 is devoted to describing how this attachment of magnetization (and the M^2RPA) fits into a Landau Fermi liquid theory formalism. In particular a new response function ($\tilde{\Pi}$) is defined that will give the self-consistent response for the magnetized quasiparticles. Section 4.1 reviews Fermi liquid theory and defines the Boltzmann equation that yields this response function as its solution. In section 4.2 we separate out the effects of the Fermi liquid coefficients that are singular in the limit $m_b \to 0$. What remains after this separation is then a Fermi liquid with reasonably weak interactions. In section 4.3 we show that approximating the response of this Fermi liquid as the response of appropriate free fermions is precisely equivalent to the M^2RPA. Finally in section 5 we make a few additional comments and summarize our findings. As an aside, in appendix A, a sum rule is derived for the Fermi liquid coefficients in the Chern–Simons Fermi liquid.

2. Review

2.1. Basics

We begin by considering a system of N interacting spin-polarized (or spinless) electrons of band mass m_b in a magnetic field $B = \nabla \times \boldsymbol{A}$. The Hamiltonian for this system is written as

$$H = \sum_j \frac{\left[\boldsymbol{p}_j - (e/c)\boldsymbol{A}(\boldsymbol{r}_j)\right]^2}{2m_b} + \sum_{i<j} v(\boldsymbol{r}_i - \boldsymbol{r}_j) \tag{1}$$

where v is the two-body interaction potential, c is the speed of light and e is the charge of the electron. We will often specialize to the physical case of Coulombic interaction $v(r) = e^2/(\epsilon r)$ with ϵ the background dielectric function. However, it will also be useful at times to consider other forms of electron–electron interaction.

Ignoring interactions between the electrons, the single-particle spectrum breaks up into Landau levels with energy $E_n = \hbar\omega_c(n + \frac{1}{2})$ where

$$\omega_c = \frac{eB}{m_b c} \tag{2}$$

is the cyclotron frequency. Each such Landau band has a degeneracy of B/ϕ_0 per unit area where

$$\phi_0 = \frac{2\pi\hbar c}{e} \tag{3}$$

is the flux quantum. The filling fraction

$$\nu = \frac{\phi_0 n_e}{B} \tag{4}$$

where is the n_e the electron density thus gives the number of Landau levels completely filled. Note that when an integer number of Landau bands are completely filled (i.e., ν is an integer), there is a discontinuity in the chemical potential leading to an incompressible integer quantized Hall state [12].

When ν is a fraction (particularly for $\nu < 1$), due to the degeneracy of single-particle states, the physics is controlled by the inter-electron interaction. We note that the interaction energy scale is given by $v(l_0)$ where $l_0 = \sqrt{\phi_0/(2\pi B)}$ is the magnetic length. In the large-magnetic-field limit (or equivalently when $m_b \to 0$), the interaction energy scale is much less than the cyclotron scale. However, due to the large degeneracy of states, traditional perturbation methods in terms of the interaction v are not effective for $\nu < 1$. In order to understand this regime, we will use the Chern–Simons transformation described below.

2.2. Chern–Simons transformation

Writing the electron wavefunction $\Phi(z_1, z_2, \ldots, z_N)$ with $z_j = x_j + \mathrm{i}y_j$ the position of the jth electron, it can be shown that [5, 1] if Φ is a solution of the Schrödinger equation $H\Phi = E\Phi$, then for m an integer,

$$\Psi(z_1, z_2, \ldots, z_N) = \prod_{i<j}\left[\frac{(z_i - z_j)}{|z_i - z_j|}\right]^{2m} \Phi(z_1, z_2, \ldots, z_N) \tag{5}$$

is a solution to the Schrödinger equation $H'\Psi = E\Psi$ with

$$H' = \sum_j \frac{\left[\boldsymbol{p}_j - (e/c)\boldsymbol{A}(\boldsymbol{r}_j) + (e/c)\boldsymbol{a}(\boldsymbol{r}_j)\right]^2}{2m_b} + \sum_{i<j} v(\boldsymbol{r}_i - \boldsymbol{r}_j) \tag{6}$$

the Hamiltonian for N interacting fermions where $\boldsymbol{a}$ is the 'Chern–Simons' vector potential

$$\boldsymbol{a}(\boldsymbol{r}) = \frac{\tilde{\phi}\phi_0}{2\pi}\sum_{j=1}^{N} \frac{\hat{\boldsymbol{z}} \times (\boldsymbol{r} - \boldsymbol{r}_j)}{|\boldsymbol{r} - \boldsymbol{r}_j|^2} \tag{7}$$

and $\tilde{\phi} = 2m$. The Chern–Simons magnetic field $b(\boldsymbol{r})$ associated with the vector potential $\boldsymbol{a}$ is given by

$$b(\boldsymbol{r}) = \boldsymbol{\nabla} \times \boldsymbol{a}(\boldsymbol{r}) = \tilde{\phi}\phi_0 \sum_{j=1}^{N} \delta(\boldsymbol{r} - \boldsymbol{r}_j) = n(\boldsymbol{r})\tilde{\phi}\phi_0 \tag{8}$$

where $n(\boldsymbol{r})$ is the local particle density. In other words, the Chern–Simons transformation can be described as the exact modelling of an electron as a fermion attached to $\tilde{\phi} = 2m$ flux quanta. We call these fermions 'gauge-transformed', 'composite', or 'Chern–Simons' fermions†.

2.3. Mean-field theory

The simplest approach to analysing this system is to make the mean-field approximation in which density is assumed uniform and the Chern–Simons flux quanta attached to the fermions are smeared out into a uniform magnetic field of magnitude

$$\langle b \rangle = n_e \tilde{\phi} \phi_0 \tag{9}$$

with n_e the average density, and $\tilde{\phi} = 2m$ again. Choosing the Chern–Simons flux to be in the opposite direction to the applied magnetic field, at some special value of the filling fraction, when $B = \langle b \rangle$, the applied magnetic field precisely cancels the Chern–Simons flux at the mean-field level. This exact cancellation occurs at the filling fraction

$$\nu = \frac{n_e \phi_0}{\langle b \rangle} = \frac{1}{2m}. \tag{10}$$

At these special filling fractions, the mean-field system can be described as fermions in zero magnetic field, and should therefore be a compressible Fermi-liquid-like state. The existence of this Fermi-liquid-like state at even-denominator filling fractions was predicted by Kalmeyer and Zhang [6] and by Halperin, Lee, and Read [1]. It should be noted that this mean-field description of the $\nu = 1/2m$ state is a nondegenerate starting point for attempting a controlled perturbation theory—unlike the original highly degenerate Landau levels.

For completeness, we also consider the case where the filling fraction is away from $\nu = 1/2m$. Here, the applied magnetic field and the Chern–Simons flux do not cancel. At the mean-field level, a residual field

$$\Delta B = B - \langle b \rangle = B - \tilde{\phi} n_e \phi_0 = B - 2mn_e \phi_0 \tag{11}$$

is left over. Thus, the mean-field system is described as noninteracting fermions in the uniform field ΔB. The effective filling fraction for these gauge-transformed fermions is given by

$$p = \frac{n_e \phi_0}{\Delta B}. \tag{12}$$

When p is a small integer, at the mean-field level, this is just a system of $|p|$ filled Landau levels of fermions, and one should observe the integer quantized Hall effect of transformed fermions. Using equation (11) as well as the definition of the filling fraction (equation (4)), this condition (equation (12)) yields precisely the Jain series [3] of fractional quantized Hall states

$$\nu = \frac{p}{2mp + 1.} \tag{13}$$

Thus, the fractional quantized Hall effect at these filling fractions is identified with an integer quantized Hall effect of gauge-transformed fermions [5]. The excitation gaps for

† Note that term 'composite fermion' is used by Jain [3] in a somewhat different sense.

these quantized Hall states are naturally given by the corresponding effective cyclotron frequency of the composite fermions

$$E_g = \hbar\,\Delta\omega_c^* = \frac{\hbar e\,\Delta B}{m^*_{gap}(\nu)c} \tag{14}$$

where $m^*_{gap}(\nu)$ is an effective mass to be discussed further below.

2.4. Perturbative approaches

Although at a mean-field level, the $\nu = 1/2m$ system looks like a Fermi liquid, we do not expect such a simple mean-field approximation to accurately describe the system. Previous attempts for going beyond mean-field theory have so far involved perturbative treatments of the Chern–Simons and electrostatic interactions [1, 7–10, 13]. There are several major difficulties in these approaches. To begin with, the 'small' dimensionless parameter that one must use in the perturbation theory is $\tilde{\phi} = 2m \geqslant 2$ which is by no means small. So although the mean-field solution seems like a good starting point for a controlled perturbation theory, the remaining interactions are quite strong and are not in the perturbative regime.

Furthermore, even if $\tilde{\phi}$ were small there would still be problems with the perturbative treatment of the Chern–Simons theory†. One problem that has attracted much attention arises when the electrostatic interaction $v(r)$ is of Coulomb form or is shorter ranged. If this is the case, it is found that composite fermion's effective mass at the Fermi surface diverges, due to infra-red gauge-field fluctuations [1, 7, 8]. Although this divergence is reflected in the energy gaps (see equation (14)) of fractional quantized Hall states at $\nu = p/(2mp + 1)$ (for large p) [7], the diverging effective mass is thought not to affect the electronic linear response at $\nu = 1/2m$ at zero temperature, due to a mutual cancellation with another singular term [7–10]. Consequently [7] the low-energy excitations at $\nu = 1/2m$ are best characterized by another, *finite*, effective mass, denoted by m^*, which is the effective mass of relevance to the present work. It is this m^* which should determine the scale of the fractional Hall gaps for small values of p.

In order to avoid the complications associated with this divergence, we can consider in this paper a system with interactions that are longer ranged than Coulomb such that there are no infra-red divergences. (The long-range interaction suppresses density fluctuations and hence kills the effects of the gauge field at long distances.) However, due to the above mentioned cancellation of divergences in physical response functions [7–10], we believe that the conclusions reached below will be independent of the range of the interaction.

In sections 3 and 4 below we will address a completely independent problem that occurs in the limit of $m_b \to 0$ (or equivalently for large magnetic field B). In this limit the ground state and low-energy excitations are constrained to the lowest Landau level. This led to restrictions on the electromagnetic response that are not properly described by simple perturbative approaches. In section 2.5 below, we will define this response function, and in section 2.6 we will describe the simplest approaches for going beyond mean field—the RPA and MRPA approximations. Finally, in section 3 we will show why these approximations are are insufficient in the $m_b \to 0$ limit.

† Perturbing in $\tilde{\phi}$ can be considered appropriate for the modelling of a system of *anyons* with statistical angle θ in a magnetic field $B = \theta n\phi_0/(2\pi)$ (here fermions are defined to have statistical angle 0 modulo 2π). By similarly attaching $\tilde{\phi} = \theta/\pi$ quanta of flux to each particle, we obtain a system that in mean-field theory is described as fermions in zero field. This family of anyonic systems with different θ-parameters presumably share many similar properties. So long as no phase transitions occur between $\theta = 0$ and $\theta = 2\pi$, the properties of the composite fermion system ($\theta = 2\pi$ or $\tilde{\phi} = 2$) should be qualitatively described by perturbation theory in $\tilde{\phi}$.

2.5. Response functions

The quantity that we will attempt to calculate is the electromagnetic response matrix $K_{\mu\nu}$ which is closely related to the conductivity [1, 13] (see equations (20), (21), and (24) below). To define K, a weak vector potential $A_\mu^{\rm ext}$ is externally applied to a system at wavevector $\boldsymbol{q}$ and frequency ω, and consequently, a current j_μ is induced (here A_0 is the scalar potential, and j_0 is the induced density). We write the response function in the form

$$j_\mu(q,\omega) = K_{\mu\nu}(q,\omega)A_\nu^{\rm ext}(q,\omega) \tag{15}$$

where μ and ν take the values $0, x, y$. We will use the convention that the perturbation is applied with $\boldsymbol{q}\|\hat{\boldsymbol{x}}$ so that the longitudinal current is $j_x = (\omega/q)\,j_0$. Using the gauge $\boldsymbol{A}_x = 0$, we can then treat $K_{\mu\nu}$ as a 2×2 matrix with indices taking the values 0 or 1 denoting the time or transverse space components. In this notation the current vector j_μ is (j_0, j_y), and the vector potential A_μ is (A_0, A_y). Note that from here on, we will routinely drop the explicit matrix subscripts μ and ν as well as the explicit q- and ω-dependences.

In systems with long-ranged Coulomb interactions, a density $j_0(q)$ induced by the external vector potential gives rise to an additional Coulomb scalar potential $ev(q)j_0(q)$, where $v(q) = 2\pi/\epsilon q$ is the Fourier transform of the usual Coulomb interaction $v(r) = 1/\epsilon r$ (with ϵ the background dielectric constant). Similarly, for the Chern–Simons fermion theory of the $\nu = 1/2m$ state, in addition, an induced vector potential originates from the composite fermions' flux. An excess density $j_0(q)$ carries an excess flux $2\pi\tilde{\phi}j_0(q)$ with $\tilde{\phi} = 2m$. A composite fermions' current $\boldsymbol{j}(q)$ is a current of flux tubes, inducing an electric field $2\pi\tilde{\phi}\boldsymbol{j}(q)$. Thus, the composite fermions' current induces also a vector potential. Keeping a matrix notation, we may write the induced vector potential as

$$A^{\rm ind} = Uj \tag{16}$$

where

$$U = \begin{bmatrix} v(q) & 0 \\ 0 & 0 \end{bmatrix} + \frac{2\pi\tilde{\phi}\hbar}{e}\begin{bmatrix} 0 & -{\rm i}/q \\ {\rm i}/q & 0 \end{bmatrix} \tag{17}$$

where the first term is the Coulomb contribution and the second term is the Chern–Simons contribution. (We have now dropped the explicit q- and ω-dependences as well as the matrix subscripts in equation (16).)

Above, we have discussed the electromagnetic response function K which gives the current response to the externally applied vector potential. It is now useful to define another response function Π, which relates the current j_μ to the *total* vector potential†,

$$j = \Pi A^{\rm total} \tag{18}$$

with

$$A^{\rm total} = A^{\rm ext} + A^{\rm ind} \tag{19}$$

so that

$$K^{-1} = \Pi^{-1} + U. \tag{20}$$

Thus Π is the part of K that is irreducible with respect to both Coulomb and Chern–Simons interactions.

† Our matrix Π is written as $\tilde{K}$ in references [1] and [13]. However, our notation for Π agrees with that used in references [7], [8] and [11].

The matrix Π also defines the finite frequency and wavevector composite fermion resistivity† ρ_{cf} via

$$\rho_{cf} = [T\Pi T]^{-1} \tag{21}$$

where T is the conversion matrix

$$T = \begin{bmatrix} \mathrm{i}\sqrt{\mathrm{i}\omega}/q & 0 \\ 0 & 1/\sqrt{\mathrm{i}\omega} \end{bmatrix}. \tag{22}$$

The composite fermion resistivity ρ_{cf} is the matrix that relates the $\hat{\boldsymbol{x}}$- and $\hat{\boldsymbol{y}}$-components of the total (induced and external) electric field $\boldsymbol{E}^{\text{total}}$ to the $\hat{\boldsymbol{x}}$- and $\hat{\boldsymbol{y}}$-components of the current $\boldsymbol{j}$ via the 2×2 matrix equation

$$\boldsymbol{E}^{\text{total}} = \rho_{cf}\boldsymbol{j} \tag{23}$$

where $\boldsymbol{E}^{\text{total}}$ is the electric field associated with the vector potential A^{total}. Equation (21) simply converts Π to ρ_{cf} by using appropriate factors of ω and q to convert $\boldsymbol{E}^{\text{total}}$ to A^{total}, and j_0 to j_x.

In terms of this composite fermion resistivity, the original electron resistivity ρ (at finite q and ω) is given by [14, 6, 1]

$$\rho = \rho_{cf} + \rho_{CS} \tag{24}$$

with

$$\rho_{CS} = \frac{2\pi\hbar\tilde{\phi}}{e^2}\begin{bmatrix} 0 & 1 \\ -1 & 0 \end{bmatrix}. \tag{25}$$

2.6. RPA and modified RPA (MRPA)

In order to find the electromagnetic response K at even a crude level, we must account for the interactions (both Coulomb and Chern–Simons) beyond mean field. The simplest approach to account for these interactions is the random-phase approximation (RPA). Making the separation of K into Π and U as described above in equation (20), the RPA approximation is equivalent to approximating Π as the response K^0 of noninteracting electrons of mass m_b in the (mean) uniform magnetic field ΔB. Such an approximation was originally discussed by Lopez and Fradkin for the Jain series of fractional quantized Hall states [5] and by Halperin, Lee, and Read [1] and Kalmeyer and Zhang [6] for the even-denominator states. In terms of resistivities, the RPA amounts to defining the composite fermion resistivity ρ_{cf} to be the resistivity for a system of free fermions with mass m_b.

As pointed out in reference [1], if one makes this RPA approximation and in the calculation of K^0 one uses the bare band mass m_b, then, at least at mean-field level, it is this mass that determines the scale of the low-energy excitations (i.e., $m^*_{gap}(\nu) = m_b$ in equation (14)). Since the low-energy excitations should be controlled by the interaction strength, this is clearly incorrect. Of course, if one could properly treat the fluctuations of the gauge field, presumably the scale of the low-energy excitations would indeed be found to be on the interaction scale‡. We note, however, that at the present no approximation is known that properly achieves the low-energy excitation scale by including fluctuations. Thus, a realistic approximation must have this low-energy excitation scale repaired by hand.

† In references [1] and [13] ρ_{cf} is called $\tilde{\rho}$.

‡ Note that in the Chern–Simons boson model of the fractional quantized Hall effect, properly treating the vortex configurations of the superfluid, can be shown to give the low-energy excitations correctly on the interaction scale [15].

The simplest way to repair the problem of having low-energy excitations on the wrong energy scale is to phenomenologically approximate Π as K^{0*}, the response of a system of noninteracting electrons in the mean magnetic field ΔB with a new effective mass m^*, where m^* is some phenomenological effective mass set by the interaction scale [1] (so $1/m^* \sim e^2/(\epsilon l_0)$). For typical experimental parameters, the measured effective mass is of the order of 4 to 15 times that of the bare band mass [1, 2]. Unfortunately, simply replacing m_b by m^* leads to a theory with several serious problems. The strategy that we will generally employ is to adopt this mass replacement, identify the resulting problems and find ways to repair them phenomenologically. Once again we note that if we had a way to properly treat the gauge-field fluctuations such that the low-energy excitations were naturally on the interaction scale, we would not have the problems that we will discuss and attempt to repair below.

To begin with, it can be shown that the naive replacement of m_b by m^* results in a theory that violates Galilean invariance [13]. In particular, Kohn's theorem (a result of Galilean invariance) requires that the only excitation mode with weight in the long-wavelength limit is the cyclotron mode at frequency $\omega_c = eB/m_b c$. This mode is a reflection of the oscillation of the centre of mass of the entire system and must therefore be independent of interactions. If one naively replaces m_b by m^*, once ends up with a cyclotron mode instead at the incorrect renormalized cyclotron frequency eB/m^*c. Similarly, simply replacing m_b by m^* results in a violation of the so-called f-sum rule [13]. We will show later in section 3 that this replacement of the band mass with the effective mass has a number of additional effects that need to be properly treated before we obtain a fully viable phenomenological theory.

In reference [13] a Modified RPA (MRPA) was constructed that restores Galilean invariance while keeping the low-energy excitations on the interaction scale. In this MRPA approximation, the mass renormalization from m_b to m^* is compensated for by including a Fermi liquid interaction coefficient $\mathcal{F}_1$ (this will be discussed further below). To define the MRPA, we write

$$\Pi^{-1} = [\Pi^*]^{-1} + \mathcal{F}_1 \tag{26}$$

$$\mathcal{F}_1 = \frac{(m^* - m_b)}{n_e e^2} \begin{pmatrix} \omega^2/q^2 & 0 \\ 0 & -1 \end{pmatrix}. \tag{27}$$

The MRPA is then obtained by setting Π^* equal to the response K^{0*} of a system of noninteracting fermions of mass m^* in the mean magnetic field ΔB. The response function thus calculated (using $\Pi^* = K^{0*}$ and equations (26) and (20)) will be called K^{MRPA}. Note that the form of equation (26) is similar to that of equation (20) in the sense that it separates out the effect of an interaction term. Similarly to the case of the RPA approach of equation (20), here $\mathcal{F}_1$ is an effective interaction and Π^* is a response function calculated without the interaction $\mathcal{F}_1$ included. Comparisons of results of exact diagonalizations of small systems projected to the lowest Landau level to results of K_{00} calculated in the MRPA were quite favourable [16] for the low-energy excitations at $\nu = p/(2mp+1)$ for small p. Similar comparisons at $\nu = \frac{1}{2}$ also yielded favourable results for small systems [17]. Despite these successes, we will show below that the (M)RPA does not properly represent the other elements of the response matrix (K_{01}, K_{10}, and K_{11}) in the limit of $m_b \to 0$.

3. Magnetized fermions

We now turn to consider the limit of small band mass m_b (or equivalently large magnetic field B). The fact that, in this limit, the electronic ground state and low-energy excitations are constrained to the lowest Landau level, leads to certain features of the electronic response

to an external static vector potential which are not properly represented in approximation schemes such as the mean field or the (M)RPA if we have used a renormalized mass m^* to achieve the correct energy scale for low-energy excitations. We note that this problem occurs in the Chern–Simons theory even when gauge-field fluctuations are not infra-red singular. (For example, if the electron–electron repulsion falls off more slowly than $1/r$ there should be no infra-red divergences in the effective mass.)

In reference [11], a new approach is proposed that is based on a separation of the current into a magnetization current which is associated with the cyclotron motion of electrons and a transport current associated with the guiding centre motion. This separation is achieved by attaching a magnetization μ_{M} to each particle. This magnetization originates from the electrons' orbital motion and is unrelated to the spin (we have assumed spinless electrons throughout this paper). In the limit $m_b \to 0$, the magnetization μ_{M} is given by the Bohr magneton

$$\mu_b = \frac{e\hbar}{2m_b c}. \tag{28}$$

The proposed separation procedure combined with approximations similar to those made in the MRPA results in an approximation that we call the M^{2}RPA that yields response functions that correctly describe the $m_b \to 0$ limit.

3.1. Zero-frequency response

In this section we shall examine the form of the zero-frequency finite-wavevector response in the high-magnetic-field (or $m_b \to 0$) limit. An acceptable approximation for calculating the response of the $\nu = 1/2m$ state must correctly predict this limit. We will show below that the usual Chern–Simons approaches do not correctly predict this limit. We then discuss in section 3.3 below how the magnetization attachment proposed in reference [11] corrects this problem.

Consider the $\nu = 1/2m$ state in the limit $m_b \to 0$. In this limit the gap between Landau levels becomes large so we expect such a system to be restricted to the lowest Landau level. If we apply a weak external static scalar potential at wavevector q to the system, the resulting state should remain in the lowest Landau level so the induced density fluctuation should depend only on the interaction strength, and not on the bare mass m_b. Thus, K_{00}, the so-called density–density response, should be independent of the bare mass in this limit (or more properly, should scale as $(m_b)^0$ plus $\mathcal{O}(m_b)$ corrections). However, the resulting density inhomogeneity will yield a transverse current called the magnetization current, given by (here and below the speed of light $c = 1$)

$$\boldsymbol{j}_{\mathrm{mag}} = \hat{\boldsymbol{z}} \times \boldsymbol{\nabla} \boldsymbol{M} \tag{29}$$

with $\boldsymbol{M}$ the magnetization density. For noninteracting particles in the lowest Landau level, the kinetic energy density is

$$E \equiv \boldsymbol{M} \cdot \boldsymbol{B} = \frac{1}{2}\hbar\omega_c n_e \tag{30}$$

so that the magnetization per particle is $|\boldsymbol{M}|/n_e = \mu_b$, the Bohr magneton. More generally, when interactions are taken into account, we let the magnetization per particle be given by a quantity μ_{M} which must become μ_b in the $m_b \to 0$ limit where the system becomes

projected to the lowest Landau level. We can thus write [18] the magnetization current as†

$$\boldsymbol{j}_{\mathrm{mag}} = \mu_{\mathrm{M}}(\hat{\boldsymbol{z}} \times \boldsymbol{\nabla} n) \tag{31}$$

with $n(\boldsymbol{r})$ the local electron density. The physical interpretation of this magnetization current is as follows. Each particle in the lowest Landau level can be thought of as a particle in a cyclotron orbit. When the density of particles is uniform, the local currents of all of these orbits cancel and there is no net current in the system. However, when there is a density inhomogeneity, these local currents do not quite cancel and a net magnetization current results. Note that this magnetization current associated with density gradients can be modelled by imagining that a small magnetization μ_{M} (equivalent to a current loop) is attached to each quasiparticle.

Using equation (31) we see that in the limit $m_b \to 0$, when we apply the weak static scalar potential $A_0^{\mathrm{ext}}(q)$ to the system and we look at the leading current response we find a magnetization current $\mu_b \hat{\boldsymbol{z}} \times \mathrm{i}\boldsymbol{q} K_{00} A_0^{\mathrm{ext}}$. Thus, if q is finite we expect

$$\lim_{m_b \to 0} K_{10}/K_{00} = \mathrm{i}q\mu_b. \tag{32}$$

This result is not contained in works based on the Chern–Simons approach previous to that of reference [11].

We can also consider applying a weak external transverse vector potential A_1^{ext} at wavevector q and zero frequency. This transverse field generates a magnetic field $\delta B = \mathrm{i}qA_1$ at wavevector q. The variation in the total magnetic field $B(\boldsymbol{r}) = B_{1/2} + \delta B(\boldsymbol{r})$ will make the kinetic energy $\frac{1}{2}\hbar\omega_c(\boldsymbol{r}) = \mu_b B(\boldsymbol{r})$ positionally dependent thus attracting electrons to the regions of minimal magnetic field when $m_b \to 0$. This attraction is not modelled in the Chern–Simons fermion picture at the mean-field or (M)RPA level if a renormalized mass is used.

Formally, if the applied variation in magnetic field generates a density fluctuation $j_0(q)$, we can write the energy cost as

$$\delta E = j_0(\delta B)\mu_{\mathrm{M}} + \frac{1}{2} K_{00} j_0^2 \tag{33}$$

where K_{00} is independent of m_b as discussed above. The first term here is just the change in local cyclotron energy which can be thought of as an effective scalar potential for the fermions. This term would occur quite naturally if we were to imagine that a magnetization μ_{M} were attached to each fermion. The second term in equation (33) is due to the Coulomb interactions within the lowest Landau level. Again note that μ_{M} must become μ_b in the $m_b \to 0$ limit, but more generally can include pieces on the interaction scale.

Minimizing the energy (equation (33)) with respect to j_0 yields the density

$$j_0 = -(\delta B)\mu_m K_{00} = -\mathrm{i}q\mu_{\mathrm{M}} K_{00} A_1 \tag{34}$$

from which we conclude that that the leading term of K_{01} is given by $\mathrm{i}q\mu_{\mathrm{M}}K_{00}$ (in accordance with the symmetry requirement of the matrix K).

Finally, once we have determined the density fluctuation due to this local magnetic field fluctuation, we again realize that this density fluctuation results in a magnetization current, so we have a leading piece of K_{11} given by $K_{00}q^2\mu_{\mathrm{M}}^2$.

† When projected to the lowest Landau level, the projected current and density operators satisfy $P\boldsymbol{j}P = \mu_b(\hat{\boldsymbol{z}} \times \boldsymbol{\nabla} P n P)$ where P is the projection operator. In other words, for projected states, all of the current is magnetization current.

3.2. Binding magnetization to composite fermions

As suggested by the above discussion, the necessary correction to the composite fermion picture involves attaching a magnetization μ_{M} to each composite fermion so that it properly represents a particle in the lowest Landau level. Attaching magnetization to each particle can also be interpreted as attaching a current loop to each particle associated with the electrons' cyclotron motion. Thus the total current would include both a piece from the motion of the particle–current loop composite and a piece from the current loop itself. To this end, we define a transport current†

$$\boldsymbol{j}_{\mathrm{trans}} = \boldsymbol{j}_{\mathrm{total}} - \boldsymbol{j}_{\mathrm{mag}} \tag{35}$$

which is the current of magnetized gauge-transformed fermions, whereas the magnetization current, as discussed above (see equation (31)) is the current associated with the attached current loops.

In addition, particles bound to magnetization should experience an effective potential associated with any local changes in the magnetic field. Thus we define the effective scalar potential

$$A_0^{\mathrm{eff}} = A_0 + \mu_{\mathrm{M}}\,\delta B. \tag{36}$$

This interaction of the bound magnetization with the magnetic field should be thought of as the effective potential associated with the local change in the cyclotron energy.

If we keep the conventions that all perturbations are applied with $\boldsymbol{q}\|\hat{\boldsymbol{x}}$, and use the Coulomb gauge again, we can rewrite equations (35) and (36) as

$$j_{\mathrm{total}} = M\, j_{\mathrm{trans}} \tag{37}$$

$$A_{\mathrm{eff}} = M^{\dagger} A \tag{38}$$

where

$$M = \begin{bmatrix} 1 & 0 \\ \mathrm{i}q\mu_M & 1 \end{bmatrix}. \tag{39}$$

In these equations, all currents are written as two vectors (j_0, j_y) and vector potentials are written as two vectors (A_0, A_y). The matrix M should be thought of as an operator that attaches magnetization. As discussed above, in the limit $m_b \to 0$, we must have $\mu_{\mathrm{M}} \to \mu_b$ in the matrix M, but more generally we can allow corrections on the interaction scale. In the rest of this paper, however, we will focus on the $m_b \to 0$ limit and consider $\mu_{\mathrm{M}} = \mu_b$.

3.3. Magnetized modified RPA (M^2RPA)

As discussed above, the (M)RPA approach does not properly model the magnetization effects discussed in section 3.1. This error is presumably due to the fact that when we take the mass-renormalized mean-field solution as a starting point for a perturbation theory for the Chern–Simons fermions, we lose the fact that the original electrons travel in local cyclotron orbits. In the approach discussed here [11], we will recover this physics by artificially attaching magnetization to each particle by hand. This attachment is not an exact transformation, but is rather a way of modelling behaviour that is lost when we take the mean field as a starting point. However, as we will see below, within a Landau–Fermi liquid theory picture, this attachment seems to give the correct quasiparticles for the system.

† The division into $\boldsymbol{j}_{\mathrm{trans}}$ and $\boldsymbol{j}_{\mathrm{mag}}$ has some degree of arbitrariness. Note that the definitions in the present paper allow for a nonzero transverse component of $\boldsymbol{j}_{\mathrm{trans}}$ in equilibrium for an inhomogeneous interacting electron system.

The magnetized particles have the same interactions (U) as the particles in the traditional Chern–Simons fermion picture. However, here, the magnetized fermions now respond to the effective potential and the motion of these magnetized fermions yields only the transport current response. We thus define a matrix $\tilde{K}$ to be the *transport* current response of the electrons to the external *effective* potential. In other words,

$$K = M\tilde{K}M^{\dagger}. \tag{40}$$

The 'magnetized modified RPA' or M^{2}RPA is then defined by setting $\tilde{K}$ equal to K^{MRPA}. Thus we have

$$K^{\mathrm{M^2RPA}} = MK^{\mathrm{MRPA}}M^{\dagger} = M([K^{0*}]^{-1} + \mathcal{F}_1 + U)^{-1}M^{\dagger}. \tag{41}$$

It should be noted that

$$K_{00}^{\mathrm{M^2RPA}} = K_{00}^{\mathrm{MRPA}} \tag{42}$$

and therefore the exact diagonalizations [16] that agreed well with calculations of K_{00} in the MRPA agree equally well with predictions of the M^{2}RPA. However, the MRPA and M^{2}RPA differ at finite q in their predictions for the other elements of the matrix K. For example,

$$K_{10}^{\mathrm{M^2RPA}} = K_{10}^{\mathrm{MRPA}} + \mathrm{i}q\mu_{\mathrm{M}}K_{00}^{\mathrm{MRPA}}. \tag{43}$$

It should be noted however, that all finite-q experimental tests [2] of the Chern–Simons theory to date have measured only K_{00} and therefore do not distinguish between the MRPA and the M^{2}RPA. As required, in the limit $m_b \to 0$, the M^{2}RPA correctly describes the static response properties described above. For example, equation (43) clearly satisfies equation (32).

As is the case for the MRPA, we expect that the M^{2}RPA, in addition to describing the $\nu = 1/2m$ Fermi liquid states, should properly describe the Jain series of quantized states $\nu = p/(2mp+1)$ for small p. At large values of p, in the case of Coulomb interactions, the description should be modified to account for the effects of the singular infra-red gauge fluctuations. In particular, the excitations at high q are sensitive to the infra-red divergence of the effective mass due to the gauge-field fluctuations [7, 8] which are neglected in the M^{2}RPA.

4. Fermi liquid theory

We now turn to discussing how the M^{2}RPA fits into the general picture of a Fermi liquid theory of the $\nu = 1/2m$ state. In essence, we will show that M^{2}RPA roughly amounts to adopting the Fermi liquid picture of reference [7] as describing the dynamics of magnetized composite fermion quasiparticles rather than unmagnetized ones.

In Landau Fermi liquid theory for fermions with short-ranged interactions, such as ^{3}He, the response function K is given by the solution of a Landau–Boltzmann equation [19, 20] which describes the dynamics of quasiparticles near the Fermi surface. In such an approach, the quasiparticles are characterized by their effective mass, m^*, and by the Landau interaction function, $f(\boldsymbol{k}, \boldsymbol{k}')$, describing the *short-range* interaction between quasiparticles of momenta $\boldsymbol{k}$ and $\boldsymbol{k}'$. In the case of ^{3}He, the quasiparticle effective mass is approximately three times the bare mass, such that the quasiparticle is quite different from the original particle. In our composite fermion system, our quasiparticle will not only have a renormalized mass, but also a renormalized magnetization.

For fermions with long-ranged interactions [19, 20], the Silin extension of the Landau theory asserts that it is the polarization Π that is described by the Landau–Boltzmann

equation (see equation (20)) rather than the full response K. In other words, equation (20) separates out the Hartree part of the long-ranged interaction such that Π gives the quasiparticle response to the sum of the external vector potential and the induced internal vector potential. The Landau–Silin approach has been very successful for the description of electrons in metals [19, 20, 21] (where there is only a long-ranged Coulomb interaction and no Chern–Simons interaction). There, Π is calculated with a Boltzmann equation describing the dynamics of quasiparticles of mass m^* interacting via a *residual* short-ranged interaction $f(\boldsymbol{k}, \boldsymbol{k}')$. Here we will try to construct a similar Landau–Silin theory for the magnetized quasiparticles in the Chern–Simons theory.

For the Chern–Simons theory, in addition to separating the long-ranged part of the interaction U, for the magnetized fermions, further separation should be carried out to remove the magnetization effects. To this end we define a response function $\tilde{\Pi}$ by

$$\Pi = M\tilde{\Pi}M^{\dagger}. \tag{44}$$

By definition, $\tilde{\Pi}$ relates the transport current of the *magnetized* quasiparticles to the *effective* total vector potential, including both external and internally induced contributions (see equations (37), (38) and (19)). For the Chern–Simons system it is $\tilde{\Pi}$ which we claim is given by a Landau–Boltzmann equation describing the dynamics of quasiparticles with the finite effective mass m^* interacting via a residual short-ranged interaction $f(\boldsymbol{k}, \boldsymbol{k}')$.

4.1. Boltzmann transport

In the Chern–Simons Fermi liquid, as in traditional Fermi liquid theory, the (magnetized) quasiparticles are characterized by their effective mass, m^*, and by the short-ranged Landau interaction function, $f(\boldsymbol{k}, \boldsymbol{k}')$. Since $|k| \approx |k'| \approx k_{\mathrm{F}}$, where k_{F} is the Fermi momentum, f is mostly† a function of θ, the angle between $\boldsymbol{k}$ and $\boldsymbol{k}'$. It is often more convenient to work with the Fourier-transformed quantity

$$f_l = \frac{1}{2\pi}\int_0^{2\pi} \mathrm{d}\theta \; f(\theta)\mathrm{e}^{\mathrm{i}l\theta}. \tag{45}$$

Due to the symmetry of the interaction function $f(\theta) = f(2\pi - \theta)$ we expect that $f_l = f_{-l}$.

In order to calculate the response function $\tilde{\Pi}$, we keep with the convention that the driving force $\boldsymbol{F}$ is applied with wavevector $\boldsymbol{q}\|\hat{\boldsymbol{x}}$, and at frequency ω (i.e., the perturbation is proportional to $\mathrm{e}^{\mathrm{i}qx-\mathrm{i}\omega t}$). Writing the fluctuations of the Fermi surface as $\delta n(\boldsymbol{p}) = \nu(\theta)\,\delta(|\boldsymbol{p}| - p_{\mathrm{F}})$ where θ is the direction of $\boldsymbol{p}$ on the Fermi surface‡, the Boltzmann transport equation can be written as [19, 20, 21]

$$-\mathrm{i}\omega\nu(\theta) + \mathrm{i}q v_{\mathrm{F}}^* \cos(\theta)[\nu(\theta) + \delta\epsilon_1(\theta)] = \boldsymbol{F}\cdot\hat{\boldsymbol{n}}(\theta) \tag{46}$$

where $v_{\mathrm{F}}^* = p_{\mathrm{F}}/m^*$ is the mass renormalized Fermi velocity,

$$\delta\epsilon_1(\theta) = \frac{m^*}{(2\pi\hbar)^2}\int \mathrm{d}\theta' \; f(\theta - \theta')\nu(\theta') \tag{47}$$

and the directional vector is given by

$$\hat{\boldsymbol{n}}(\theta) = (\cos\theta, \sin\theta). \tag{48}$$

† In the case of Coulomb or shorter-ranged inter-electron interactions, perturbative approaches [7] find that f may have a singular dependence on $|k|$. One hopes that in a fully renormalized theory (nonperturbatively) these singularities do not prevent us from writing a Boltzmann transport equation. We note that Kim *et al* [8] recently showed that a form of quantum Boltzmann equation can be derived that is independent of these singularities.

‡ The definition of ν agrees with that in references [7], [19], and [20] but differs from the function f used in references [21] and [13] by a factor of v_{F}^*.

Equation (46) is just the usual Boltzmann equation of Fermi liquid theory. However, here the driving force is given by the total *effective* electric field

$$\boldsymbol{F} = -e\boldsymbol{E}_{\text{eff}}^{\text{total}} = -e\left(\boldsymbol{\nabla} A_{\text{eff }0}^{\text{total}} - \frac{\mathrm{d}}{\mathrm{d}t}\boldsymbol{A}_{\text{eff}}^{\text{total}}\right). \tag{49}$$

where (see equations (19) and (38))

$$A_{\text{eff}}^{\text{total}} = M^{\dagger} A^{\text{total}}. \tag{50}$$

Once one has solved equation (46) for $\nu(\theta)$, The local charge density can be written as the density of quasiparticles [19, 20, 21]

$$j_0 = \frac{-ep_{\text{F}}}{(2\pi\hbar)^2}\int \mathrm{d}\theta\ \nu(\theta). \tag{51}$$

Similarly, the motion of these magnetized quasiparticles gives the local *transport* current density

$$\boldsymbol{j}_{\text{trans}} = \frac{1}{m^*}\left[\frac{-ep_{\text{F}}^2}{(2\pi\hbar)^2}\right]\int \mathrm{d}\theta\ \hat{\boldsymbol{n}}(\theta)\{\nu(\theta) + \delta\epsilon_1(\theta)\} \tag{52}$$

or

$$\boldsymbol{j}_{\text{trans}} = \frac{1}{m_b}\left[\frac{-en_e}{\pi}\right]\int \mathrm{d}\theta\ \hat{\boldsymbol{n}}(\theta)\nu(\theta). \tag{53}$$

Thus one can easily find the magnetized quasiparticle resistivity matrix $\tilde{\rho}_{cf}$ relating the effective electric field to the transport current via the 2×2 matrix equation (cf. equation (23))

$$\boldsymbol{E}_{\text{eff}}^{\text{total}} = \tilde{\rho}_{cf}\boldsymbol{j}_{\text{trans}}. \tag{54}$$

The response matrix $\tilde{\Pi}$ is then given by (cf. equation (21))

$$\tilde{\Pi} = [T\tilde{\rho}_{cf}T]^{-1}. \tag{55}$$

We now have a prescription for calculating the response K of the Chern–Simons Fermi liquid given the effective mass m^* and the interaction function $f(\theta)$. To reiterate, the prescription is to solve the Boltzmann equation (equation (46)) for $\nu(\theta)$ and calculate the current using equation (52) to get the magnetized composite fermion resistivity $\tilde{\rho}_{cf}$. The response K can then be obtained by using equations (55), (44) and (20).

4.2. Separating singular Fermi liquid coefficients

As discussed above, one expects that the effective mass, which determines the energy scale of the low-energy excitations, should be set by the Coulomb interaction scale. Similarly, one expects [7] that the interaction function $f(\theta)$ should be on the interaction scale (i.e., proportional to $1/m^*$). However, two important restrictions on f yield pieces of f that are set by the larger scale $1/m_b$.

A well known result of Fermi liquid theory [19, 20] is that the Fermi liquid coefficients f_0 and f_1 are fixed by the identities

$$\frac{1}{m_b} = \frac{1}{m^*} + \frac{f_1}{2\pi\hbar^2} \tag{56}$$

$$\frac{\mathrm{d}\mu}{\mathrm{d}n} = \frac{2\pi\hbar^2}{m^*} + f_0. \tag{57}$$

The identity (56) is a result of Galilean invariance [19, 20]. (Note that f_1 refers to the first Fourier mode of excitations of the Fermi surface which corresponds to a Galilean boost.)

Thus, f_1 is clearly on the larger scale $1/m_b$ rather than the interaction scale. Furthermore, we claim that the sum rule (57) fixes f_0 to be on the scale $1/m_b$ also. This counterintuitive result is due to the fact that the compressibility derivative $\mathrm{d}\mu/\mathrm{d}n$ is taken at fixed ΔB. One can understand this [7, 8] by realizing that the Fermi liquid theory uses the mean-field zero-effective-field solution for its ground state. When a particle is added or subtracted, in order to maintain a Fermi liquid (i.e., zero effective field), the external field must increased by $\tilde{\phi}$ flux quanta to compensate for the added Chern–Simons field. Thus, at fixed $\Delta B = 0$, the magnetic field is linked to the density n via $B = \tilde{\phi} n \Phi_0$. In the limit $m_b \to 0$, the interaction energy between the magnetization $\boldsymbol{M} = \mu_b n$ and the external field is given by

$$E = \boldsymbol{M} \cdot \boldsymbol{B} = \frac{\pi \tilde{\phi} \overline{h}^2 n^2}{m_b}. \tag{58}$$

Of course this can also be thought of as the cyclotron energy. Differentiating this with respect to n we obtain a magnetization contribution to the chemical potential

$$\mu^{\mathrm{mag}} = \frac{2\pi \tilde{\phi} \overline{h}^2 n}{m_b} = \overline{h}\omega_c \tag{59}$$

such that the magnetization contribution $\tilde{f}_0$ to the zeroth Fermi liquid coefficient f_0 is given by

$$\tilde{f}_0 = \frac{\mathrm{d}\mu^{\mathrm{mag}}}{\mathrm{d}n} = \frac{2\pi \tilde{\phi} \overline{h}^2}{m_b} \tag{60}$$

which is also the inverse compressibility of free electrons of mass m_b at constant ΔB. The coefficient f_0 is written as $f_0 = \tilde{f}_0 + \delta f_0$ where $\tilde{f}_0$ is $\mathcal{O}(m_b^{-1})$ and δf_0 is on the smaller interaction scale. As mentioned in reference [7], in the limit $m_b \to 0$, the requirement that the low-energy spectrum is independent of m_b forces the other interaction coefficients (f_l for $l \neq 0, 1$) to be on the interaction scale. In addition we note that using the Pauli exclusion principle a sum rule can be derived for the remaining Fermi liquid coefficients f_l for $l \neq 0, 1$. This sum rule is derived explicitly in appendix A.

Since in the limit of $m_b \to 0$, $\tilde{f}_0$ and f_1 are on the bare mass scale whereas all other coefficients f_l (as well as δf_0) are expected to be on the smaller interaction scale, we will separate out the contributions of these two coefficients by writing

$$\tilde{\Pi}^{-1} = [\tilde{\Pi}^*]^{-1} + \tilde{\mathcal{F}}_0 + \mathcal{F}_1 \tag{61}$$

where

$$\tilde{\mathcal{F}}_0 = \begin{pmatrix} \tilde{f}_0 & 0 \\ 0 & 0 \end{pmatrix} \tag{62}$$

and $\mathcal{F}_1$ is given by equation (27). The function $\tilde{\Pi}^*$ is to be calculated using a Landau–Boltzmann equation representing quasiparticles with the same effective mass m^* and interaction coefficients f_l except that f_1 is artificially set to zero and the magnetic contribution $\tilde{f}_0$ is subtracted off f_0. Once again, the form of equation (61) looks like the form of equation (20) where we have separated two interaction terms and defined the remaining response $\tilde{\Pi}^*$ to be the response of a similar Fermi liquid with those interactions removed. The separation of the coefficient f_0, analogous to taking $v(q) \to v(q) + f_0$ in equation (20), is justified by noting that f_0 corresponds to a short-ranged density–density interaction. Similarly, the separation of the coefficient f_1 is achieved by noting that the f_1 coefficient corresponds to a current–current interaction ($\mathcal{F}_1$) which can similarly be added on in equation (61). The separation of the nonzero f_1 coefficient [13] is analogous to that described in equation (26) (the coefficient of the matrix in equation (27) is proportional to

f_1) and is derived explicitly in reference [13]. Note that the separation of the effects of Fermi liquid coefficients by treating them as density–density and current–current interactions can only be done for f_0 and f_1 and not for any f_l for $l > 1$. Having made this separation, we expect that the response $\tilde{\Pi}^*(q, \omega)$ is independent of m_b in the limit $m_b \to 0$ and is well behaved for all values of q/m_b. The transformation equations (20), (26), (27), (44), and (61) do not in themselves involve any approximations, and may be considered simply as a means of defining a new 'irreducible' response function $\tilde{\Pi}^*(q, \omega)$.

4.3. Relation to the M^2RPA

To relate this Fermi liquid approach to the M^{2}RPA we note the identity

$$U + \tilde{\mathcal{F}}_0 = M^{\dagger -1} U M^{-1} \tag{63}$$

which holds in the limit $m_b \to 0$. This identity is a statement of the fact that if you allow the magnetization to see the Chern–Simons magnetic field as well as the external magnetic field, then the $1/m_b$ contribution to f_0 will vanish since the magnetization now sees zero magnetic field on average. We will also need

$$\tilde{\mathcal{F}}_0 = M^{\dagger} \tilde{\mathcal{F}}_0 M \tag{64}$$

which is just the statement that a density–density interaction does not care whether or not the particles are magnetized. Using these identities, we find that M^{2}RPA defined in equation (41) is equivalent to approximating Π^* by K^{0*}, the response of a free Fermi gas of particles of mass m^*, and calculating the response using equations (20), (44), and (61).

We note that in Fermi liquid theory, the Landau–Boltzmann equation does not correctly describe the Landau diamagnetic contribution to the transverse static response. Similarly, we suspect that here the function $\tilde{\Pi}^*_{11}$ derived from the Landau–Boltzmann equation lacks a term of the form $q^2\chi$ where χ is some appropriate Landau susceptibility which we expect to be on the scale of the interaction strength. As usual, if we fix the ratio ω/q to be nonzero, and take $q \to 0$, this diamagnetic term becomes negligible. However, when $\tilde{\Pi}^*$ is approximated as K^{0*} for the M^{2}RPA, this diamagnetic contribution is included at least approximately.

5. Further comments and conclusions

5.1. The effect of other Fermi liquid coefficients

Clearly, the M^{2}RPA involves neglecting Fermi liquid coefficients f_l for $l \neq 0, 1$. Although this formally violates the sum rule of appendix A, the neglect of these interaction terms is probably quite reasonable. In previously studied Fermi liquid theories (helium-3 and electrons in metals) although the first few Fermi liquid coefficients may be large, the higher ones become rapidly smaller [20].

To elucidate the effects of additional nonzero Fermi liquid coefficients, we consider the addition of a nonzero magnetic field ΔB. At the Jain series of filling fractions $p/(2mp + 1)$, the composite fermions fill precisely p Landau levels, resulting in fractionally quantized states. The Boltzmann excitation spectrum for composite fermions for these states [13, 21] is given by

$$\omega_n = n\left(1 + \frac{m^* f_n}{2\pi\hbar^2}\right)\Delta\omega_c^* \tag{65}$$

where $\Delta\omega_c^* = e\,\Delta B/m^*$, and n is a positive integer. The residue (or weight) of the nth excitation mode is proportional to q^{2n}. Since only the $n = 1$ mode has weight in the small-q limit, this is the only mode that is altered by the Chern–Simons or Coulomb interactions (equation (20)). Thus, the response spectrum (i.e., the location of poles of K_{00}) is identical to the composite fermion spectrum predicted by equation (65) except that the $n = 1$ mode is pushed up to the cyclotron frequency (see equation (56)) as required by Kohn's theorem [5, 13].

The $q \to 0$ spectrum shown by equation (65) would suggest that it would be very easy to extract the value of the Fermi liquid coefficients f_l from the response of the system. However, we point out that the spectrum predicted by the above Fermi liquid theory (or by the MRPA and M^{2}RPA) yields a spectrum of single-quasiparticle excitations only. This single-particle excitation spectrum should be correct at low frequency, but at higher frequency one can create multiple low-energy excitations. At least under some conditions, at finite ΔB, these multiple excitations may have more weight than the single-particle excitations in the $q \to 0$ limit [22], making it more difficult to accurately extract Fermi liquid coefficients directly from an excitation spectrum using equation (65).

5.2. Connection with other recent work

Using the M^{2}RPA approach, we can calculate the response K and hence Π (equation (20)) and hence the composite fermion conductivity $\sigma_{cf} = [\rho_{cf}]^{-1}$ via equation (21). This could equivalently be calculated by using equations (61) and (44) along with the approximation $\tilde{\Pi}^* = K^{0*}$ which, as discussed above, is equivalent to the M^{2}RPA. Either approach yields the limiting low-frequency and low-wavevector composite fermion Hall conductivity for small m_b

$$\lim_{q\to 0}\lim_{\omega\to 0}[\sigma_{cf}]_{xy} = -\left(\frac{1}{2\tilde{\phi}}\right)\left(\frac{e^2}{2\pi\hbar}\right). \tag{66}$$

For $\nu = \frac{1}{2}$, this is precisely half the value found in reference [23] *in the opposite order of limit and in the presence of disorder*. Since these two results are for slightly different cases, it is not clear that there is any contradiction. Reference [23] also calculates the above order of limits for a clean system and finds it to be zero to first order in perturbation theory in $\tilde{\phi}$. Although the calculations described in reference [23] would be a natural direction for attempting to understand this attachment, at lowest order the magnetization effects are not seen. Once again this result does not directly contradict our work since it is only perturbative. Note that our result, being inversely proportional to $\tilde{\phi}$, may indicate why the perturbative approach yields zero.

5.3. Possible relation to other pictures of quantum Hall states

Since much of our knowledge of quantum Hall states stems from the use of trial wavefunctions [3, 12, 25], it is natural to try to make contact with these approaches. Typically the trial wavefunctions are projected to the lowest Landau level which in some senses can be thought of as the $m_b \to 0$ limit†. Since the magnetization attachment described in this paper is concerned with exactly this limit, it is interesting to see to what extent the physics described in this paper matches the physics described by the trial wavefunctions. Particularly interesting would be a comparison to the predictions of the $\nu = \frac{1}{2}$ lowest Landau level wavefunction constructed by Haldane [25].

† Note that projection and $m_b \to 0$ are not formally equivalent. See for example reference [24].

5.4. Conclusions

The M^{2}RPA describes the $\nu = 1/2m$ state as a Fermi liquid of magnetized composite fermions with a finite renormalized effective mass m^*, an f_1-parameter dictated by Galilean invariance and an f_0-parameter originating from the interaction of the magnetization with the magnetic field. All remaining Fermi liquid parameters (which are expected to be on the much smaller interaction scale) are neglected. The M^{2}RPA predicts the same K_{00} as the MRPA, but in contrast it yields the correct behaviour for K_{01}, K_{10}, and K_{11} in the limit $m_b \to 0$ for arbitrarily small q.

By separating the pieces due to magnetization and due to singular Fermi liquid coefficients (in the $m_b \to 0$ limit) we identify a response function $\tilde{\Pi}^*$ that is represented by the solution of a well behaved Landau–Boltzmann equation (up to diamagnetic terms). Our claim that $\tilde{\Pi}^*$ is well behaved in the limit $m_b \to 0$ we believe to be an exact statement (although we have not proved it rigorously) independent of the approximation used to define the M^{2}RPA.

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Appendix A. The Fermi liquid sum rule

By using the Pauli exclusion principle for the forward-scattering amplitude, a sum rule can be derived for the Landau coefficients for a Fermi liquid [26]. In two dimensions for a spinless Fermi liquid, the form of this sum rule is (in this appendix $\hbar = e = c = 1$)

$$\sum_{l=-\infty}^{\infty} \frac{f_l}{(2\pi/m^*) + f_l} = 0. \tag{A1}$$

In reference [27] this sum rule is generalized to the case of Landau–Silin Fermi liquid theory for systems with long-ranged Coulomb interactions. In two dimensions, for a spinless Fermi liquid with long-range interactions, the sum rule then becomes

$$\sum_{\substack{l=-\infty \\ l\neq 0}}^{\infty} \frac{f_l}{(2\pi/m^*) + f_l} = -1. \tag{A2}$$

In this appendix we will derive the form of the sum rule in two dimensions for a spinless two-dimensional Fermi liquid interacting via a long-ranged Chern–Simons gauge field as well as via a direct 'Coulomb' interaction $v(r)$. As much as possible, we will use the notation of references [19] and [27]. In this derivation, we will assume for simplicity that there are no complications due to infra-red divergent gauge-field fluctuations. This should be rigorously true in the case where the interaction is longer ranged than the Coulomb one. One hopes that in the case of Coulomb and shorter-ranged interactions, cancellation of divergences similar to those found in the calculations of the electromagnetic response [8, 9, 10] will lead to a fully renormalized theory that also obeys the sum rule derived

here. Note that a formal derivation of the sum rule is given first, followed by a simple phenomenological interpretation.

As in reference [27] we write the full vertex function ${}^0\Gamma$ in terms of 'proper' or irreducible four-point function $\tilde{\Gamma}$ by writing

$$ {}^0\Gamma(p, p'; \bar{\omega}) = {}^0\tilde{\Gamma}(p, p'; \bar{\omega}) + \tilde{\Lambda}(p, \bar{\omega}) \left[1 - U(q)\tilde{S}(\bar{\omega})\right]^{-1} U(q)\tilde{\Lambda}(p', \bar{\omega}) \tag{A3} $$

where $p = (\omega, \boldsymbol{k})$, $p' = (\omega', \boldsymbol{k}')$ and $\bar{\omega} = (\epsilon, \boldsymbol{q}) = p' - p$. Note that all three vectors will be arranged such that the zeroth element of the vector p is the frequency element (this differs from the notation of references [19] and [27]). Here, the three-vector $\tilde{\Lambda}$ is the proper three-point vertex function, and the three-by-three matrix $\tilde{S}$ is the proper polarization propagator. Also, we have the interaction matrix

$$ U(q) = \frac{-\mathrm{i}}{2\pi} \begin{pmatrix} v(q) & 0 & \mathrm{i}c(q) \\ 0 & 0 & 0 \\ -\mathrm{i}c(q) & 0 & 0 \end{pmatrix} \tag{A4} $$

with $c(q) = 2\pi\tilde{\phi}/q$ the Chern–Simons gauge interaction [1], and $v(q)$ the direct Coulomb interaction. As mentioned above, although the physical case is $v(q) = 2\pi/(\epsilon q)$, we may want to consider other functional forms. Note that the zeroth row and column of the matrix U represent the interactions of the density whereas the first and second row and column represent the longitudinal and transverse current respectively. Equation (A3) is shown diagrammatically in figure A1.

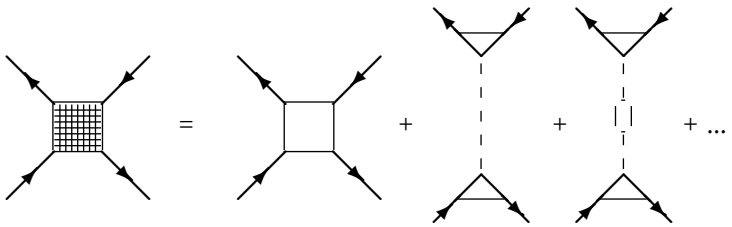

Figure A1. Separation of the four-point function ${}^0\Gamma$ into its irreducible part ${}^0\tilde{\Gamma}$ and self-consistent-field interaction contributions. This is the diagrammatic representation of equation (A3). Here the dotted line is the interaction propagator U that includes both Coulomb and Chern–Simons terms.

The Landau interaction function for a Landau–Silin Fermi liquid theory is given in terms of the proper four-point function by [19, 27]

$$ f(\boldsymbol{k}, \boldsymbol{k}') = 2\pi\mathrm{i} z_k z_{k'} \lim_{q/\epsilon \to 0} \lim_{\bar{\omega} \to 0} {}^0\tilde{\Gamma}(p, p'; \bar{\omega}) \tag{A5} $$

or

$$ f(\boldsymbol{k}, \boldsymbol{k}') = 2\pi\mathrm{i} z_k z_{k'} {}^0\tilde{\Gamma}^0(\boldsymbol{k}, \boldsymbol{k}') \tag{A6} $$

where k and k' are taken on the Fermi surface, and z_k is the quasiparticle renormalization.

The Pauli principle, on the other hand, dictates that [27]

$$ \lim_{\bar{\omega} \to 0} \overset{0}{\lim_{(q/\epsilon) \to \infty}} \Gamma(p, p'; p - p') = {}^0\,\Gamma^{\infty}(\boldsymbol{k}, \boldsymbol{k}') = 0. \tag{A7} $$

Applying the same limits to equation (A3) we obtain

$$ {}^0\tilde{\Gamma}^{\infty}(\boldsymbol{k},\boldsymbol{k}) = \lim_{\bar{\omega}\to 0}\lim_{(q/\epsilon)\to\infty} {}^0\tilde{\Gamma}(p,p';\bar{\omega}) = \tilde{\Lambda}^{\infty}(p,\bar{\omega})\left[1+U(q)\tilde{S}^{\infty}(\bar{\omega})\right]^{-1} U(q)\tilde{\Lambda}^{\infty}(p,\bar{\omega}) \tag{A8} $$

where $\tilde{\Lambda}^{\infty}$ and $\tilde{S}^{\infty}$ are the corresponding limits of $\tilde{\Lambda}$ and $\tilde{S}$.

Ward identities [19] can be invoked to yield

$$ \tilde{\Lambda}^{\infty}(p,\bar{\omega}) = \frac{v_{\mathrm{F}}^*}{z_k}\begin{pmatrix} \partial k_{\mathrm{F}}/\partial\mu \\ 0 \\ 1 \end{pmatrix} \tag{A9} $$

where $v_{\mathrm{F}}^* = k_{\mathrm{F}}/m^*$ is the Fermi velocity, and μ is the chemical potential. It should be noted that the longitudinal current element vanishes because in the limit that $\bar{\omega}\to 0$ and $(q/\epsilon)\to\infty$ we must have $q \perp k$. Ward identities can also be used to calculate the matrix [19]

$$ \tilde{S}^{\infty} = -2\pi\mathrm{i}\ \mathrm{diag}\left[\frac{\partial n}{\partial\mu}, \frac{n}{m_b}, \frac{n}{m_b}\right] \tag{A10} $$

where m_b is the bare band mass and n is the density. Using these relations in equation (A8) we find

$$ g(\boldsymbol{k},\boldsymbol{k}) = 2\pi\mathrm{i} z_k z_k {}^0\tilde{\Gamma}^{\infty}(\boldsymbol{k},\boldsymbol{k}) \tag{A11} $$

or

$$ g(\boldsymbol{k},\boldsymbol{k}) = -v_{\mathrm{F}}^{*2}\left[(\partial k_{\mathrm{F}}/\partial\mu)^2/(\partial n/\partial\mu) + m_b/n\right]. \tag{A12} $$

Note that this relation holds for all interactions $v(q)\sim q^{-\alpha}$ with $\alpha < 2$.

Using the relation [19, 27]

$$ g(\boldsymbol{k},\boldsymbol{k}') = f(\boldsymbol{k},\boldsymbol{k}') - \int \frac{\mathrm{d}\boldsymbol{k}''}{(2\pi)^2} f(\boldsymbol{k},\boldsymbol{k}'')g(\boldsymbol{k}'',\boldsymbol{k}')\,\delta(\epsilon_{k''}-\mu) \tag{A13} $$

we derive

$$ g_l = \frac{1}{2\pi}\int_0^{2\pi} \mathrm{d}\theta\ g(\theta)\mathrm{e}^{\mathrm{i}l\theta} = \frac{f_l}{(2\pi/m^*)+f_l} \tag{A14} $$

and thus

$$ g(\theta=0) = -v_{\mathrm{F}}^{*2}\left[(\partial k_{\mathrm{F}}/\partial\mu)^2(\partial n/\partial\mu) + m_b/n\right] \tag{A15} $$

or

$$ g(\theta=0) = \sum_{l=-\infty}^{\infty}\frac{f_l}{(2\pi/m^*)+f_l}. \tag{A16} $$

Finally, using the identities (57) and (56) along with $k_{\mathrm{F}}^2 = 4\pi n$ and $v_{\mathrm{F}}^* = k_{\mathrm{F}}/m^*$ we obtain

$$ \sum_{\substack{l=-\infty \\ l\neq 0,-1,1}}^{\infty}\frac{f_l}{(2\pi/m^*)+f_l} = -3. \tag{A17} $$

Finally, we note that simple phenomenological derivations can be given for the sum rules (A2) and (A17). Since f_0 is a local density–density interaction, the inclusion of the Coulomb interaction is in some sense equivalent to taking $f_0 \to f_0 + v(q)$. As we take q to zero, this translates to the effective divergence of f_0. If we allow f_0 to diverge in sum rule equation (A1) we immediately obtain equation (A2).

The derivation for equation (A17) proceeds along a similar line. The Chern–Simons interaction as written in equation (17) is an interaction between transverse current and density. Using current conservation $\omega j_0 = q j_x$ we can write this as an interaction between the transverse and the longitudinal currents (here $\hat{x}$ is chosen to be the longitudinal direction). We thus rewrite this interaction energy as

$$\delta E = j^* U j = \frac{2\pi \tilde{\mathrm{i}}\phi}{\omega} \left[j_x^* j_y - j_y^* j_x \right]. \tag{A18}$$

Such an interaction term can be represented in Fermi liquid theory by letting $f_1 \rightarrow f_1 + 2\pi \mathrm{i}\tilde{\phi}/\omega$ and $f_{-1} \rightarrow f_{-1} - 2\pi \mathrm{i}\tilde{\phi}/\omega$. Note that f_1 being complex is a reflection of the fact that the Chern–Simons interaction is not time-reversal invariant. Letting ω and q both go to zero then leads to the effective divergence of f_0, f_1 and f_{-1}, thus yielding equation (A17) from equation (A1).

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